Macroeconomics II

Calculation of growth rates: a note

1. Discrete time-path of a variable: discrete growth rate

Let X be a variable with economic meaning, which assumes positive (and only positive) values that may change through time.

Let time be represented by a series of time periods: 0, 1, ..., T. Then, we can describe the time path of this variable by a discrete-time model. The series

(1) $x_0, x_1,..., x_T$

describe the discrete-time path of variable X.

The discrete growth rate of this variable in period t (in relation to t-1) is the value r_t such that:

(2) $x_t = x_{t-1} . (1+r_t), t = 0, ..., T$

what means that:

(3) $\boxed{r_t = (x_t / x_{t-1}) - 1 = (x_t - x_{t-1})/x_{t-1} = \Delta x_t/x_{t-1}}$

The value of r_t can be positive, nul or negative.

2. Average growth rate (discrete)

Let us consider the whole period from t=0 up to t=T.

If r remains constant through time, that is, $r_t = r$, for t = 0, 1, ..., T, then:

(4) $x_t = x_{t-1}.(1+r)$ for all t = 0, ..., T

which implies:

(5) $x_t = (1+r)^T.x_0$

But this is unusual. The growth rates $r_1, r_2, ..., r_T$ may be different, and we may be interested to know the average growth rate in the whole period from 0 to T. The average growth rate is the rate r* such that:

(6) $x_t = (1+r^*)^T.x_0$

From (6) above we get:

(7) $\ln x_T = T. \ln (1+r^*) + \ln x_0$

$\ln(1+r^*) = (\ln x_T - \ln x_0)/T$

$\ln (1+r^*) = \ln (x_T/x_0)/T$

$1+r^* = (x_T/x_0)^{1/T}$

$r^* = (x_T/x_0)^{1/T} - 1$

It is possible to prove that r* is an average of the growth rates r1, r2, ..., rT. But it is not the arithmetic average of such rates. Instead, fom (7) we get:

(8) $r^* = (x_T/x_0)^{1/T} - 1 = (x_T/x_{T-1} \,.\, x_{T-1}/x_{T-2} \,.\, \ldots.\, x_2/x_1 \,.\, x_1/x_0)^{1/T} - 1$

We may then conclude that although the average growth rate for the whole period may be calculated without information on the growth rates in the intermediate periods (see (7)), it may be expressed as the geometric average of these growth rates (see (8)).

3. Continuous-time path of a variable: instantaneous growth rate

We may describe the time-path of the variable X using a continuous-time model. That is, we may assume that time flows consinuously, and X assumes values in each moment of the whole period. Let X(t) be a continuous function of t, with a continuous first derivative em relation to t, for all t from 0 to T.

The instantanous growth rate of X at moment t is

(9) $r(t) = (d\,X(t)/d\,t)/X(t)$

that compares to (3) in the discrete-time path case.

4. How to calculate the instantaneous growth rate

From (9) we get:

(10) $r(t) = (d\,X(t)/d\,t)/X(t) = d\ln X(t)/d\,t$

Let r(t) = r, a constant through the whole period. Then, from (9) we get:

(11) $d\ln X(t)/d\,t = r$

$\ln X(t) = r.t + \text{const.}$

$X(t) = e^{r.t} \,.\, \text{const } (= X(0))$

and then:

(12) $X(t) = X(0) \,.\, e^{r.t}$

where r is the instantaneous growth rate of the variable X, assuming that it has a constinuous-time path through time, X(t) and also assuming that r(t) = r, a constant.

5. Average instantaneous growth rate (continuous)

But the general case is that the growth rate may vary throuth time, that is, r(t). If r(t) is not a constant, we may be interested to calculate the average instantaneous growth rate of the variable X thoughot the whole period, from 0 to T. This is the value r* such that:

(13) $X(t) = X(0).e^{r^*.T}$

This rate may be calculated as follows. From (13) we get:

(14) $\ln X(t) = \ln X(0) + r^*.T$

$r^* = (\ln X(t) - \ln X(0)) / T$

$\boxed{r^* = \ln (X(t)/X(0))^{1/T}}$

6. Instantaneous growth rates: properties

Let X(t) and Y(t) be two continuous variables, and K a constant. Let r(X) be the average instantaneous growth rate of X in a period and r(Y) be the average instantaneous growth rate in the same period. We may prove that:

(15a) $r(K) = 0$

(15b) $r(K.X(t)) = r(X(t))$

(15c) $r(X(t).Y(t)) = r(X(t)) + r(Y(t))$

(15d) $r(1/X(t)) = - r(X(t))$

(15e) $r(X(t)/Y(t)) = r(X(t)) - r(Y(t))$

(15f) $r(X(t)^K) = K.r(X(t))$

The strategy to prove such properties is quite straightforward. Let us prove (15c). If $X(t) = X(0).e^{r(X).t}$ and $Y(t) = Y(0).\ e^{r(Y).t}$, then $X(t).Y(t) = X(0).Y(0).e^{(r(X)+r(Y)).t}$. The average growth rate of X(t).Y(t) is then r(X)+r(Y).

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